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Ray tracing methods for wave propagation in moving plasmas

Méthodes de lancer de rayon pour la propagation d'ondes dans les plasmas en mouvement

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Abstract/*Résumé*

The propagation of a wave in a medium is in general affected when the medium is moving. Because plasma equilibria often involve plasma motion, for instance in astrophysics or for nuclear fusion, understanding the effect of motion on plasma waves is particularly important. Within this objective, we are here interested in developing a ray tracing method to study the trajectory of rays propagating in a moving plasma under the geometrical optics approximation. An effective dispersion relation for the moving medium as seen from the laboratory is first found by performing a Lorentz transformation of the dispersion relation known for the medium at rest, and then used in the ray tracing equations which give the trajectories of the rays. We find that the ordinary mode of the magnetized cold plasma is unaffected by the motion, but that the extraordinary mode undergoes a drag which increases with the velocity and which can be significant at low frequency.

La propagation d'une onde dans un milieu est en général modifiée lorsque celui-ci est en mouvement. Les configurations d'équilibre d'un plasma reposant souvent sur un champ de vitesse, comme par exemple en astrophysique ou pour la fusion par confinement magnétique, une compréhension des effets du mouvement sur les ondes plasmas est particulièrement souhaitable. On s'intéresse ici à développer une méthode de lancer de rayon pour étudier la trajectoire des rayons se propageant dans un plasma en mouvement dans l'approximation de l'optique géométrique. Une relation de dispersion effective pour le milieu en mouvement vu du laboratoire est identifiée en effectuant une transformation de Lorentz de la relation de dispersion du milieu au repos, nous permettant alors d'obtenir les équations de lancer de rayon qui donnent les trajectoires des rayons. On trouve que le mode ordinaire du plasma froid magnétisé n'est pas affecté par le mouvement du plasma, tandis que le mode extraordinaire subit un entraînement par le plasma d'autant plus grand que la vitesse est grande. Cet entraînement pourrait être important notamment à basse fréquence.

1 Context and motivations

Waves are used routinely in plasmas. Applications include plasma control, such as plasma heating for magnetic confinement fusion, as well as plasma diagnostics, such as estimating interstellar magnetic fields via Faraday rotation. The design of these control systems and the interpretation of these diagnostics rely directly on plasma waves theory to model propagation in these anisotropic dispersive media, in the presence of possible plasma non-uniformities (density, magnetic field, etc.). However, these models and the theory of waves in plasmas neglect, other than for rare exceptions, the effect of a velocity field. Indeed, even though it has long been established that motion can have an effect on wave propagation, and that rotation phenomena are encountered in a wide range of environments from laboratory plasmas to astrophysics to magnetic confinement fusion, wave propagation properties are in general determined assuming a plasma at rest. The aim of the work presented here is to present the first elements of an eikonal formalism that can subsequently be extended to numerically model a number of manifestations of the effect of motion on propagation in a plasma.

This manuscript is organized as follows. First, in Section 2, we will discuss how the methods of geometrical optics and ray tracing can be used to obtain the trajectories of waves in a moving medium. Then, in Section 3, these results will be applied to the case of a moving magnetized cold plasma, and we will examine certain effects of the motion for specific modes. Finally, some concluding remarks will be given in Section 4.

2 Geometrical optics applied to moving media

The aim of this section is to present how the classical geometrical optics (GO) methods can be applied to model the propagation of waves in moving media. Media considered here are arbitrary, e.g. they can be anisotropic and inhomogeneous. The velocity field of the medium is taken to be an arbitrary smooth vector field. The space and time variations of the inhomogeneities are required to be slow in the GO approximation. First, a brief summary of classical GO methods is given.

Then, elements on electromagnetism and wave propagation in a uniformly moving medium are introduced to identify an effective dispersion relation for the moving medium as seen from the laboratory frame. Finally, conditions under which the ray tracing equations can be applied to an arbitrarily moving medium, as well as the effect of motion, are examined.

2.1 Basics of geometrical optics

A thorough introduction to geometrical optics can be found in a number of books and papers [1–3]. Here we simply recall the key elements that we will need later to discuss ray tracing in moving media, following primarily Tracy's textbook [1].

The physics problem of interest here is the propagation of waves in a linear medium. The multicomponent wave field is denoted $\mathbf{\Psi}$ and can be for example the electric field or a density field. Let us assume that this problem is entirely described by a linear wave equation $\underline{\widetilde{\boldsymbol{D}}}(\boldsymbol{x}, i\boldsymbol{\nabla}, -i\partial_t)\mathbf{\Psi} = \mathbf{0}$ [1][1] with $\underline{\widetilde{\boldsymbol{D}}}$ a matrix of differential operators, which we want to solve in order to describe how the waves propagate in this medium. To make it less abstract, we can consider as an illustration an homogeneous isotropic nondispersive medium with refractive index n. The wave equation for the electric field $\boldsymbol{E}$ is then $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{E} + n^2/c^2\partial_t^2\boldsymbol{E} = 0$, so that in this case $\underline{\widetilde{\boldsymbol{D}}} = \underline{\boldsymbol{\nabla}}\underline{\boldsymbol{\nabla}} + n^2/c^2\partial_t^2\underline{\mathbf{1}}$ with $\underline{\boldsymbol{\nabla}}$ the matrix associated to the curl operator.

2.1.1 Eikonal expansion and geometrical optics

Geometrical optics is a method to obtain approximate solutions to the wave equation. This method requires that the characteristic variation length L of the properties of the medium is large compared to the wavelength λ of the wave [1] [2]. As a consequence, wave optics effects such as diffraction are neglected. The benefit of this approximation is that the wave equation, which is a partial differential equation (PDE) system and hence is difficult to solve, can be reduced to a system of ordinary differential equations (ODEs). This system, which is easier to solve, is called the ray tracing equations (RTEs) [1]. The solutions of the RTEs take the form of rays which are parametric curves in phase space and can be interpreted in physical space as light rays. Precisely, in physical space, rays are the integral curves of the Poynting vector field [4], or equivalently the group velocity field, and are therefore the trajectories of the wave packets [5]. Computing a large number of rays allows one to reconstruct a wave field as an approximate solution to the wave equation [1].

The essence of this method is that it consists in a scale separation between the dynamics of the phase and of the envelope. The former is supposed to evolve faster than the latter, whose variations are of the same order as those of the properties of the medium. A parameter $\epsilon = \lambda/L$ [6] is introduced to keep track of this ordering, with $\epsilon \ll 1$. This parameter is analogous to the Planck constant $\hbar$ of quantum physics [2, 5] and is therefore sometimes called the semi-classical parameter [5]. It is also called the eikonal parameter [1] or the ordering parameter [2].

RTEs are derived by assuming a quasi-plane wave form for the wave field. This means that the wavefront varies slowly so the wave is locally plane [1]. Specifically, each propagating mode supported by the medium [1, 6] is then written as

$$\mathbf{\Psi}_m(\boldsymbol{x}, t) = A(\boldsymbol{x}) \exp\left[i(\epsilon^{-1}\Phi_0(\boldsymbol{x}, t) + \Phi_1(\boldsymbol{x}))\right] \hat{\boldsymbol{e}}(\boldsymbol{x}) \tag{1}$$

where the subscript m refers to the mode m, and A, Φ_0 and Φ_1 are three real scalar fields being respectively the amplitude, the dynamical (or eikonal) phase and the amplitude phase. $\hat{\boldsymbol{e}}$ is a complex unit vector field being the polarization of the mode. We dropped here the subscript m of the mode to which these fields are associated in order to simplify the notations. All these fields have slow variations compared to the dynamical phase Φ_0. This is modelled by the presence of ϵ^{-1} multiplying Φ_0. Eq. (1) is known as eikonal approximation [1] or WKB ansatz [2, 6]

Plugging the WKB ansatz Eq. (1) in the wave equation gives a series in powers of ϵ from which wave equations at successive orders in ϵ can be extracted. The leading order in ϵ, which corresponds to the ϵ^0 terms, only gives the trajectory of the rays in phase space, i.e. the position and the wave vector $(\boldsymbol{x}(s), \boldsymbol{k}(s))$ as functions of a parameter s parameterizing the ray [1]. It is the equivalent of the classical limit $\hbar \to 0$ in quantum physics. To obtain information on the evolution of amplitude and polarization along the ray, one needs to consider the RTEs to first order $O(\epsilon)$ [1]. First-order RTEs also give first-order corrections to the trajectory describing the bending of the rays due to polarization state through spin-orbit interactions [7], including spin-Hall effect [8] or optical Magnus effect [9]. In this work, we are interested in the trajectory to zeroth-order, and thus focus on the zeroth-order RTEs.

2.1.2 Ray tracing equations for the ray trajectory

The wave equation to zeroth-order in ϵ can be written as $\underline{\boldsymbol{D}}(\boldsymbol{x}, \boldsymbol{k}, \omega)\mathbf{\Psi} = \mathbf{0}$ where $\boldsymbol{k}$ and ω, defined as $\boldsymbol{k} \doteq -\boldsymbol{\nabla}\Phi_0$ and $\omega \doteq \partial_t\Phi_0$, are respectively the zeroth-order wave vector and angular frequency and $\underline{\boldsymbol{D}}$ is a matrix called the dispersion matrix [2]. This matrix describes how the wave behaves locally. Indeed, since it is to zeroth-order in ϵ, it does not include any derivative of the medium properties. The eigenvalues of this matrix which we write $\mathcal{D}_m$ are the zeroth-order dispersion functions, and the corresponding unit eigenvectors $\hat{\boldsymbol{e}}_m$ are the unit polarization vectors of the modes [1].

1. Rigorously, the general linear wave equation is an integrodifferential equation but can be reduced to this pseudodifferential form [1] using the Weyl symbol calculus that we do not discuss in this document. Moreover, for the sake of simplicity, we assume here that it does not depend on time t although the method would be the same.

2. If the medium were not time-stationary i.e. if the wave equation depended on time, the method would require that the characteristic time of the variations in time of the medium would be large compared to the period of the wave.

The dispersion relation of the mode m is $\mathcal{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega) = 0$ and can be viewed as the Hamilton-Jacobi equation of a particle with position $\boldsymbol{x}$ and momentum $\boldsymbol{k}$, and $\mathcal{D}_m$ the Hamiltonian [2]. Using this analogy and Hamilton's equations, one writes the equations for the trajectory in phase space as [1]

$$\frac{d\boldsymbol{x}}{ds} = -\boldsymbol{\nabla}_{\boldsymbol{k}} \mathcal{D}_m, \qquad \frac{d\boldsymbol{k}}{ds} = \boldsymbol{\nabla}_{\boldsymbol{x}} \mathcal{D}_m. \tag{2}$$

We call these equations the trajectory RTEs and their solutions are the trajectories of the rays in phase space $(\boldsymbol{x}(s), \boldsymbol{k}(s))$ [1]. These trajectories are those of the energy of the wave, i.e. of the wave packet or, with a quantum physics point of view, those of the photons. These equations are similar to those of the dynamics of a point-particle [10]. The group velocity $\boldsymbol{v}_g \doteq d\boldsymbol{x}/dt$ is tangent to the ray in physical space and is the velocity of the trajectory associated to t as the ray parameter [3]. Since these equations are to zeroth-order in ϵ, they do not provide any information on the evolution of the amplitude or of the polarization.

Note importantly here that the use of these equations for a mode m only requires to have the dispersion function $\mathcal{D}_m$ describing this mode. In the next paragraph we will therefore aim at obtaining these functions, playing the role of Hamiltonians, for modes propagating in moving media.

2.2 Electromagnetism in uniformly moving media

Let us now illustrate how to relate the properties of wave propagation in a medium in uniform linear motion to a laboratory frame dispersion function describing the local behaviour of the wave.

2.2.1 Lorentz transformations

We first define two reference frames : the frame attached to the laboratory, called the lab-frame and denoted Σ, where the observer is at rest, and the frame attached to the uniformly moving medium, called the rest-frame and denoted Σ', where the medium is at rest [11, 12]. All quantities expressed in the rest-frame will be indicated by a prime. Let us also write $\boldsymbol{v}$ the constant velocity of the medium.

If we assume the lab-frame Σ to be inertial then Σ' is also inertial and the Lorentz transformations [4] can be used to transform the position four-vector $x^\mu = (ct, \boldsymbol{x})$ [5] and the wave four-vector $k^\mu = (\omega/c, \boldsymbol{k})$ from one frame to the other. The Lorentz transformations for these four-vectors are [13]

$$ct' = \gamma(ct - \boldsymbol{\beta} \cdot \boldsymbol{x}) \qquad \omega' = \gamma(\omega - c\boldsymbol{\beta} \cdot \boldsymbol{k}) \tag{3a}$$

$$\boldsymbol{x}' = (\underline{\mathbf{1}} + (\gamma - 1)\boldsymbol{\beta} \otimes \boldsymbol{\beta})\boldsymbol{x} - \gamma\boldsymbol{\beta}ct, \qquad \boldsymbol{k}' = (\underline{\mathbf{1}} + (\gamma - 1)\boldsymbol{\beta} \otimes \boldsymbol{\beta})\boldsymbol{k} - \gamma\boldsymbol{\beta}\omega/c \tag{3b}$$

where the vector $\boldsymbol{\beta} = \boldsymbol{v}/c$ is the dimensionless velocity and $\gamma = 1/\sqrt{1 - \beta^2}$ is the Lorentz factor. The first equation Eq. (3a) of the Lorentz transformation on the wave four-vector describes in particular the relativistic Doppler effect [13, 14] causing a shift in the frequency of the wave from one frame to the other depending on the angle between the wave vector and the velocity. The second equation Eq. (3b) describes the relativistic aberration effect, that is a change in the direction of the wave vector from one frame to the other which depends on its direction and on the frequency.

2.2.2 Effective medium

We would like to describe the propagation of waves as observed by an observer at rest in the lab-frame. The approach adopted here is to consider the moving medium as an equivalent effective medium at rest in lab-frame but with additional properties due to its motion [15, 16]. The velocity is then seen as a vector property of the effective medium in a way like, for example, the magnetic field can be considered as a vector property of the medium or the refractive index or the density as scalar properties.

If we denote $\mathcal{D}'_m$ the dispersion functions of the modes propagating in the medium when it is at rest i.e. in the frame Σ', then we let $\mathcal{D}_m$ be the dispersion functions of these same propagating modes but seen from lab-frame where the medium is moving. The $\mathcal{D}_m$ are therefore the dispersion functions describing the propagation of the modes in the effective medium [17]. From there, the effective medium can be treated as any classical medium at rest. But the effective medium is more complicated than the original medium since the velocity adds a preferred direction and the effective medium is then in general bianisotropic [13, 18] and spatially dispersive [15] even if the original medium is isotropic and nondispersive.

2.2.3 Dispersion relation in lab-frame

There are two ways [13, 15] to obtain the dispersion functions $\mathcal{D}_m$ of the effective medium that we need to use the RTEs and then to compute the trajectories of the rays.

The first one, used notably in Refs. [12, 19, 20], can be called the *lab-frame approach* since the main part of the calculations are done with a lab-frame point of view. The first step is to derive the constitutive relations of the effective

3. It is possible to parameterize the rays using t instead of s.
4. To be exact, here we use the Lorentz transformations for a Lorentz boost.
5. Using a common notation abuse, we designate here the contravariant (resp. covariant) four-vectors by the notation of their contravariant (resp. covariant) components where the Greek indices go from 0 to 3.

medium as seen in the lab-frame starting with the constitutive relations in the rest-frame, using the Lorentz transformations of the electromagnetic fields [21]. These relations combined with Maxwell's equations written in the lab-frame are finally used to obtain a wave equation. Proceeding in the framework of GO as explained above, the lab-frame dispersion matrix $\underline{\boldsymbol{D}}$ is obtained from the wave equation to zeroth-order in the small eikonal parameter ϵ, and the dispersion functions $\mathcal{D}_m$ of the modes are the eigenvalues of this matrix. This approach has however two major drawbacks. First, finding the eigenvalues of the dispersion matrix is only tractable for simple media. Second, the calculations become very complicated when one wants to work beyond the first order in β. To circumvent these difficulties we will mostly use the second approach.

The second approach, adopted in [22] and referred to as the *transformation approach*, uses the fact that the dispersion relation is Lorentz covariant, in the sense that the dispersion relation in one frame can be obtained from the one in the other frame only by Lorentz transforming the position four-vector $x^\mu = (ct, \boldsymbol{x})$ and the wave four-vector $k^\mu = (\omega/c, \boldsymbol{k})$ using the Lorentz transformations Eqs. (3). This result was demonstrated by Censor for an homogeneous anisotropic [14] or a bianisotropic [17] medium. Similarly to Refs. [14, 17], this can be written

$$\mathcal{D}_m(x^\mu, k^\mu) = \mathcal{D}'_m(x'^\mu(x^\mu), k'^\mu(k^\mu)) \tag{4}$$

where x'^μ and k'^μ are expressed as functions of x^μ and k^μ through the Lorentz transformations Eqs. (3). This amounts in fact to taking into account the relativistic Doppler and aberration effects seen by the moving medium because of its velocity. As a result, one only needs to know the dispersion function in the rest-frame $\mathcal{D}'_m$, which is often the case, to immediately obtain the dispersion function of the effective medium $\mathcal{D}_m$ in the lab-frame. The advantages of this approach compared to the first one are that it works for very general media [17], in particular for dispersive anisotropic media like magnetized plasmas [22] (see Section 3), and that it yields the dispersion functions to all orders in β. We must note here though that, as shown by the x^μ dependency in Eq. (4), we assume the covariance property to hold for inhomogeneous media (we assume $\mathcal{D}'_m$ to have slow spatial dependencies), whereas it was only demonstrated by Censor for homogeneous media [17].

Let us now illustrate this transformation approach by going back to our example of an isotropic nondispersive medium, which we now consider to be moving. The refractive index of this medium at rest is denoted n'. The dispersion function of the modes propagating in such a medium in rest-frame is well-known : $\mathcal{D}'_{\text{iso}} = n'^2\omega'^2/c^2 - k'^2$. Using the Lorentz transformations Eqs. (3), we then get the dispersion function in lab-frame [23]

$$\mathcal{D}_{\text{iso}} = \omega^2/c^2 - k^2 + (n'^2 - 1)\gamma^2(\omega/c - \boldsymbol{\beta}\cdot\boldsymbol{k})^2. \tag{5}$$

We see that $\mathcal{D}$ depends on the direction of $\boldsymbol{k}$ through the term $\boldsymbol{\beta}\cdot\boldsymbol{k}$ whereas it was not the case for $\mathcal{D}'$. Note also that for the vacuum ($n' = 1$) the dispersion functions in the two frames are the same, which is consistent with the fact that a moving vacuum remains a vacuum. This comes from the fact that the quantity $\omega'^2/c^2 - k'^2 = \omega^2/c^2 - k^2$ is a Lorentz scalar, and is thus invariant under Lorentz transformations.

Summing our findings, we found how to obtain the dispersion function $\mathcal{D}_m$ necessary to use the trajectory RTEs in the lab-frame as long as $\mathcal{D}'_m$ is provided. It is worth noting that to study all the modes propagating in a moving medium, one has to consider also the purely oscillatory modes and the evanescent modes which can become propagative from lab-frame although they do not propagate in the rest-frame [14]. We interpret this phenomenon as a consequence of the loss of simultaneity between the two reference frames according to special relativity.

2.3 Dispersion functions for arbitrarily moving media

Having shown in the previous paragraph how to obtain lab-frame trajectory RTEs for a medium in uniform motion, we would like to extend it to the case of a medium in arbitrary motion, as considered in Refs. [23, 24].

To this end let us consider a medium with an arbitrary time-stationary [6] velocity field $\boldsymbol{\beta}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})/c$. The idea explored here is to work within the GO framework presented above. To do so we require for the characteristic variation lengths of the velocity field to be large compared to the wavelengths [23, 25]. As a result $\boldsymbol{\beta}$ does not change significantly over the spatial scale of a wavelength, and the wave thus sees locally the medium as being in uniform linear motion. Then, assuming that "special relativity theory is valid locally and instantaneously" [25], the local behaviour of the wave is then described by $\mathcal{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ [5] which has been obtained using the methods for a uniform linear motion of Subsection 2.2, considering crucially the local velocity $\boldsymbol{\beta}(\boldsymbol{x})$ as a constant [23]. This way, we get $\mathcal{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ at each position $\boldsymbol{x}$, and the expression of $\mathcal{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$ does not contain derivatives of the velocity field as expected to zeroth-order in the eikonal small parameter.

2.4 Ray tracing in arbitrarily moving media

Now that we have the dispersion functions $\mathcal{D}_m$ of the modes propagating in the arbitrarily moving medium observed from lab-frame, we can finally compute the trajectory of the rays using the trajectory RTEs (2). Going back again to the example of a moving isotropic nondispersive medium with refractive index n', and limiting ourselves to first order in β for

6. The stationary hypothesis is only here to simplify the discussion by having a wave equation independent in time as in the explanation of GO at the beginning of the document.

the sake of simplicity, we find the trajectory RTEs

$$\frac{d\boldsymbol{x}}{ds} = 2\boldsymbol{k} + 2(n'^2 - 1)\frac{\omega}{c}\boldsymbol{\beta}, \tag{6a}$$

$$\frac{d\boldsymbol{k}}{ds} = -2(n'^2 - 1)\frac{\omega}{c}(\boldsymbol{k} \times (\boldsymbol{\nabla} \times \boldsymbol{\beta}) + (\boldsymbol{k} \cdot \boldsymbol{\nabla})\boldsymbol{\beta}). \tag{6b}$$

We can now discuss some effects the motion has on the propagation of the waves in light of Eqs. (6).

First, we see that the velocity appears in Eq. (6a) in the form of a term proportional to $\boldsymbol{\beta}$ on the right hand side, that adds to the term $2\boldsymbol{k}$ expected without motion. This term is responsible for altering the group velocity $\boldsymbol{v}_g \propto d\boldsymbol{x}/ds$ of the wave as seen in the lab-frame. There is now a finite angle between the wave vector $\boldsymbol{k}$ and the group velocity $\boldsymbol{v}_g$, in contrast with what is found in this medium at rest where those two vectors are aligned. We interpret this as a drag of the wave by the medium. Such drag was theorized by Player [26] and experimentally observed by Jones [27].

Second, we see that the velocity also appears in the second equation Eq. (6b), in the form of a contribution that depends on the velocity field non-uniformity. This was to be expected in that the velocity field non-uniformity leads to inhomogeneous properties for the effective medium. This non-uniformity causes a bending of the rays by affecting the evolution of $\boldsymbol{k}$ along the ray, just like an inhomogeneous refractive index $n'(\boldsymbol{x})$ would, which then carries over to the group velocity.

2.5 Relation to previous work

Let us now discuss briefly how the model for ray tracing in moving media proposed above, that is the combination of the trajectory RTEs Eqs. (2) and the lab-frame dispersion function $\mathcal{D}_m$ obtained by Lorentz transforming the rest-frame one $\mathcal{D}'_m$, relates to previous contributions on the topic.

Considering the particular case of an isotropic non-dispersive medium, Rozanov [3] derived from the wave equations on the electric and magnetic fields and from the expression of the Poynting vector a different set of RTEs. The derivation is however restricted to first order in β. The effects of the motion then appear as a perturbation term in the trajectory RTEs, which is a correction to the result for zero motion. This perturbation term can be interpreted, in the point-particle analogy, as an additional force term which deviates the trajectory of the rays. We verified that the trajectory RTEs Eqs. (2) equipped with the lab-frame dispersion function $\mathcal{D}_m$, found as explained above, are equivalent to the trajectory RTEs with the perturbation term found by Rozanov [3].

Still for isotropic non-dispersive medium, another approach that has been explored is to make use of a relativistic formalism [23, 24, 28, 29], valid this time to all orders in β. Gordon [28] rewrote the covariant constitutive relations for a moving isotropic medium into a unique constitutive relation similar to the one of an empty curved spacetime. This equivalent virtual spacetime is called optical spacetime and is equipped with an optical metric $\bar{g}$, which depends on the characteristics of the medium and on its motion. The effects of the moving medium on wave propagation in physical spacetime are the same as the ones of an empty spacetime adequately curved. Under this formalism the dispersion function writes [24, 28, 29]

$$\mathcal{D}_{\text{Gordon}} = \bar{g}^{\mu\nu}k_\mu k_\nu \qquad \text{with} \qquad \bar{g}^{\mu\nu} = g^{\mu\nu} + (n'^2 - 1)\beta^\mu\beta^\nu, \tag{7}$$

where $\bar{g}$ is the Gordon optical metric, g is the metric of the physical space and $\beta^\mu = \gamma(1, \boldsymbol{\beta})$ is the dimensionless four-velocity of the medium. We verified that this dispersion function is exactly equal to the one found above Eq. (5).

Compared to these earlier contributions, a strong advantage of the formulation proposed in this work, and principal motivation for its development, is that it can accommodate dispersive and anisotropic media, such as plasmas as considered next.

3 Ray tracing in moving plasmas

In this section, the theory established in Section 2 is applied to a moving cold plasma. First the dispersion functions for a moving cold plasma as seen from the lab-frame are derived. These expressions are then used to discuss some effects of motion on the propagation of O and X modes.

3.1 Dispersion functions for moving cold plasmas

The problem of finding the dispersion relations from the lab-frame of a plasma in uniform linear motion has already received attention, both using the lab-frame approach [12, 20, 30] and the transformation approach [22, 31]. These previous contributions did however focus primarily on the electronic response. Here we follow the second approach presented in Section 2 to rederive these results without any particular assumption. As a reminder, it consists in performing Lorentz transformations in the rest-frame dispersion functions $\mathcal{D}'_m(\boldsymbol{x}', \boldsymbol{k}', \omega')$ to obtain the lab-frame ones $\mathcal{D}_m(\boldsymbol{x}, \boldsymbol{k}, \omega)$.

3.1.1 Moving unmagnetized plasma

There are two different modes in an unmagnetized plasma at rest : the electromagnetic propagating mode and the electrostatic oscillating mode. The latter does not propagate.

The dispersion function of the propagating mode (O mode) is [32]

$$\mathcal{D}'_O(\boldsymbol{k}', \omega') = \omega'^2/c^2 - k'^2 - \omega'^2_p/c^2 \tag{8}$$

where $\omega'^2_p = \omega'^2_{pe} + \omega'^2_{pi}$ with ω'_{ps} the plasma frequency of the species s as mesured in the rest-frame Σ'. Recalling that the quantity $\omega'^2/c^2 - k'^2$ is a Lorentz scalar, the lab-frame dispersion function writes straightforwardly

$$\mathcal{D}_O(\boldsymbol{k}, \omega) = \omega^2/c^2 - k^2 - \omega'^2_p/c^2. \tag{9}$$

The dispersion function of the oscillating mode is [14]

$$\mathcal{D}'_{\text{osc}}(\omega') = \omega'^2 - \omega'^2_p. \tag{10}$$

Expressing ω' as a function of ω and $\boldsymbol{k}$ according to the Lorentz transformations Eqs. (3), we obtain lab-frame dispersion function

$$\mathcal{D}_{\text{osc}}(\boldsymbol{x}, \boldsymbol{k}, \omega) = \gamma^2(\omega - c\boldsymbol{k} \cdot \boldsymbol{\beta}(\boldsymbol{x}))^2 - \omega'^2_p \tag{11}$$

Unlike $\mathcal{D}'_{\text{osc}}$, $\mathcal{D}_{\text{osc}}$ contains both ω and $\boldsymbol{k}$ meaning that this dispersion function describes a propagative mode. The oscillating mode, which does not propagate in rest-frame, does propagate in lab-frame [14].

3.1.2 Moving magnetized plasma

The method presented in Section 2 can be applied to obtain the dispersion functions in the lab-frame for the modes propagating at an arbitrary angle with respect to the background magnetic field $\boldsymbol{B}_0$ in a moving magnetized cold plasma. This amounts to expressing from the Lorentz transformations Eqs. (3) ω' and $\boldsymbol{k}'$ in terms of ω and $\boldsymbol{k}$ in the Appleton-Hartree equation [33] that is well known in the plasma rest-frame, to obtain its equivalent in the lab-frame.

Rather than dealing with these general formulas, which are rather heavy, we focus here on the simpler case where the wave vector is perpendicular to the magnetic field ($\boldsymbol{k} \perp \boldsymbol{B}_0$). In this configuration, that is for perpendicular propagation, the modes propagating in the rest-frame are the well known O and X modes. The O mode is the same as in an unmagnetized plasma. From the discussion above we already have its dispersion function from the lab-frame, that is Eq. (9). Moving on to the X mode, the dispersion function in the rest-frame is

$$\mathcal{D}'_X(\boldsymbol{k}', \omega') = \omega'^2 - k'^2c^2 + \omega'^2 \left(\chi'_\perp(\omega') - \frac{\chi'^2_\times(\omega')}{1 + \chi'_\perp(\omega')} \right) \tag{12}$$

with $\chi'_\perp$ and $\chi'_\times$ the components of the susceptibility tensor $\underline{\chi}'$ of the magnetized cold plasma in the rest-frame. Using the Lorentz transformations Eqs. (3) it directly yields the dispersion function in the lab-frame

$$\mathcal{D}_X(\boldsymbol{x}, \boldsymbol{k}, \omega) = \omega^2 - k^2c^2 + \gamma^2(\omega - c\boldsymbol{k} \cdot \boldsymbol{\beta}(\boldsymbol{x}))^2 \left(\chi'_\perp(\gamma(\omega - c\boldsymbol{k} \cdot \boldsymbol{\beta}(\boldsymbol{x}))) - \frac{\chi'^2_\times(\gamma(\omega - c\boldsymbol{k} \cdot \boldsymbol{\beta}(\boldsymbol{x})))}{1 + \chi'_\perp(\gamma(\omega - c\boldsymbol{k} \cdot \boldsymbol{\beta}(\boldsymbol{x})))} \right). \tag{13}$$

We immediately see that $\mathcal{D}_X$ now depends on the direction of $\boldsymbol{k}$, whereas it was not the case for $\mathcal{D}'_X$.

3.2 Effects of motion on propagation of O and X modes

Having derived the lab-frame dispersion functions for O and X modes, let us now discuss some effects of the motion on these two classical modes. We focus here for the sake of simplicity on the case of a uniform motion.

3.2.1 O mode

We showed above that the dispersion function for the O mode is Lorentz invariant, meaning that it has exactly the same form whether it is expressed in Σ' or Σ [34]. In particular, the dispersion function from the lab-frame $\mathcal{D}_O$ has no dependence on the velocity $\boldsymbol{\beta}$. This is a notable feature since it means that the O mode is not affected by the motion of the plasma. In other words, it propagates in the same way whether the plasma is moving or not. As a consequence, it is not possible to probe the motion of a cold plasma using an O mode.

We note further that the plasma frequency is also Lorentz invariant, i.e. we have $\omega_p = \omega'_p$ [35] with ω_p (resp. ω'_p) the plasma frequency as measured in the frame Σ (resp. Σ'). The cutoff of the O mode is therefore unaffected by the motion.

3.2.2 X mode

Unlike the O mode, the propagation of the X mode is modified by the motion of the plasma. In particular the X mode ray is dragged in the direction of the velocity of the medium : its group velocity $\boldsymbol{v}_g$ is no longer aligned with its wave vector $\boldsymbol{k}$ due to an extra term associated with motion which depends on the velocity $\boldsymbol{\beta}$.

To illustrate this effect, we consider the case where the velocity is perpendicular to the wave vector ($\boldsymbol{v} \perp \boldsymbol{k}$). Consequently, the configuration studied here is such that the vectors ($\boldsymbol{v}, \boldsymbol{k}, \boldsymbol{B}_0$) form an orthogonal trihedron. Figure 1 shows the angle

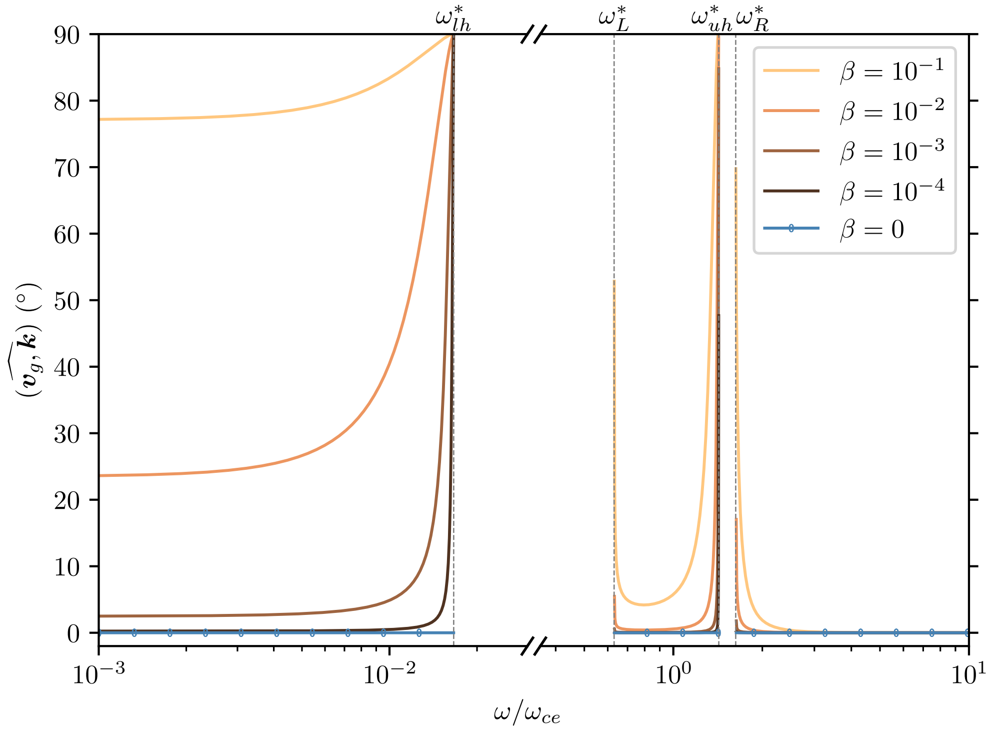

Figure 1 : Angle between the group velocity $\boldsymbol{v}_g$ and the wave vector $\boldsymbol{k}$ as a function of the frequency for an X mode in a moving plasma with $(\boldsymbol{v}, \boldsymbol{k}, \boldsymbol{B}_0)$ an orthogonal trihedron for several values of the velocity. Here the density is $n_e = 10^{19}$ m^{-3} and the magnetic field is $B_0 = 1$ T. The superscript * indicates a normalization by the electron cyclotron frequency ω_{ce}.

$(\widehat{\boldsymbol{v}_g, \boldsymbol{k}})$ between the group velocity $\boldsymbol{v}_g$ and the wave vector $\boldsymbol{k}$ as a function of the dimensionless frequency ω/ω_{ce} of the wave, with ω_{ce} the electron cyclotron frequency, and for several values of the dimensionless velocity $\beta = v/c$. We use here as a baseline a plasma density $n_e = 10^{19}$ m^{-3} and a magnetic field $B_0 = 1$ T.

Without motion, i.e. for $\beta = 0$, we verify in Figure 1 that the X mode propagates as usual, that is to say that the group velocity is then aligned with the wave vector ($\boldsymbol{v}_g \parallel \boldsymbol{k}$) so that the angle $(\widehat{\boldsymbol{v}_g, \boldsymbol{k}})$ is zero for all frequencies. Moreover, we recover the usual propagation branches of this mode with the associated lower and upper hybrid resonances ω_{lh} and ω_{uh}, and cutoffs ω_L and ω_R [32].

For finite velocity, Figure 1 shows on the other hand that the angle $(\widehat{\boldsymbol{v}_g, \boldsymbol{k}})$ is non zero and positive for all frequencies. This means that the X mode is dragged in the direction of the motion. For all frequencies, this drag increases with velocity. Looking more closely, we find that the maximum angle 90°, which represents a limit case where the wave is fully dragged by the medium, is reached at resonances and cutoffs. Yet, predictions for these frequencies are questionable as the theory is expected to break down. Indeed, on the one hand, the cold plasma model is not able to describe well the behaviour near resonances. One the other hand, at cutoffs, the wavelength becomes so large that the GO approximation no longer holds. Notwithstanding these limits, the strong drag observed in Figure 1 near cutoffs and resonances calls for refined studies.

Away from resonance and cutoffs, we find that the drag angle is small at high frequency ($\ll 1°$), short of very large velocities ($\beta \gtrsim 10^{-3}$). It goes to zero at very high frequencies since the wave does not interact with the plasma and then behaves like if it were in vacuum. On the other hand, we see that a possibly significant and constant drag is found at low frequency. This asymptotic behavior is due to the fact that much below the lower hybrid frequency, and for $v_A \ll c$ with $v_A \propto B_0/\sqrt{n_i}$ the Alfvén velocity, the plasma behaves like a non-dispersive medium with a refractive index $n' \propto c/v_A \gg 1$ [32]. Because the drag angle for the X mode then behaves as that for a non-dispersive isotropic medium, that is scaled as $n' - 1/n'$ [26], the drag coefficient is an increasing function of the refractive index. We thus expect the low frequency drag angle to increase with the plasma density n_e and to decrease with the magnetic field B_0. In particular the drag of a few degrees observed for $\beta \sim 10^{-4}$ for the plasma parameters considered here would be even larger for a denser plasma at the same field.

4 Conclusion

To conclude, after recalling some basic elements of classical geometrical optics, we showed how this method can be used to study the propagation of waves in arbitrarily moving media, in the limit that the velocity field varies slowly in space and time compared to the wavelength and the period of the wave considered. Under this assumption, the lab-frame dispersion function, that is required in the trajectory RTEs, is obtained by Lorentz transforming the dispersion function assumed to be known in the rest-frame. The trajectory RTEs can then be used to compute the trajectory of the wave similarly to a point-particle.

The trajectory RTEs give in particular access to the group velocity in the lab-frame. We find that, as a result of motion, an additional term depending on $\boldsymbol{\beta}$ appears in the lab-frame group velocity. This term captures the wave drag induced by the moving medium. Applying these results to a moving magnetized cold plasma for perpendicular propagation, one finds that the O mode is unaffected by the motion, while the X mode undergoes a drag in the direction of the velocity. This drag can be significant, especially near resonances and cutoffs, but also below the lower hybrid frequency.

Looking ahead, the work presented here focused on the ray trajectory and thus only considered the zeroth-order of the eikonal expansion. As such it cannot inform on the polarization and amplitude evolution. Capturing the effect of motion on polarization, such as polarization drag [36, 37], will require extending this work to obtain first-order RTEs. This development is currently underway.

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